LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (3/9/2017 - 3/15/2017)

Problem: Find, with proof, all continuously differentiable functions $f(x)$ of real variable x for which $f(0) = 0$ and

$$|f'(x)| \leq |f(x)| \quad \text{for all real} \quad x.$$

Solution: The only such function is the constant function $f(x) = 0$.

Consider the set $\mathcal{U} = \{x \in \mathbb{R} \mid f(x) = 0\}$; the set $\mathcal{U}$ is non-empty due to the given initial condition. Also, $\mathcal{U}$ is closed as the pre-image under the continuous function f of the closed set $\{0\}$. It remains to show that $\mathcal{U}$ is open; by connectedness of $\mathbb{R}$ we then know that $\mathcal{U} = \mathbb{R}$.

Let $x_0 \in \mathcal{U}$ and fix some $0 < \varepsilon < 1$. Let

$$M_{x_0} = \max_{x_0 - \varepsilon \leq x \leq x_0 + \varepsilon} |f(x)|. \tag{1}$$

Suppose that $M_{x_0} \neq 0$. We note that $|f'(x)| \leq |f(x)| \leq M_{x_0}$ on the interval $[x_0 - \varepsilon, x_0 + \varepsilon]$ and that – due to the Mean Value Theorem –

$$|f(x)| = |f(x) - f(x_0)| \leq M_{x_0}|x - x_0| \leq M_{x_0}\varepsilon < M_{x_0} \quad \text{for all} \quad x_0 - \varepsilon \leq x \leq x_0 + \varepsilon.$$

This contradicts the assumption (1). Thus, $M_{x_0} = 0$ and $[x_0 - \varepsilon, x_0 + \varepsilon] \subseteq \mathcal{U}$.