LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

# PUZZLE OF THE WEEK (2/23/2017 - 3/1/2017)

**Problem:** Let $n$ be a positive integer. Find, in terms of $n$, the value of the determinant of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 2 & 3 & 4 & \dots & n-1 & n & n \\ 3 & 4 & 5 & \dots & n & n & n \\ .. & .. & .. & \dots & .. & .. & .. \\ n-1 & n & n & \dots & n & n & n \\ n & n & n & \dots & n & n & n \end{pmatrix}$$

and justify your claim.

**Solution:** The value of the determinant is $(-1)^{n(n-1)/2} \cdot n$.

The value of the determinant is invariant under subtraction of one row from another. By subtracting the $(n-1)$-st row from the last row, followed by subtracting the $(n-2)$-nd row from the $(n-1)$-st row, and just generally subtracting the $i$-th row from the $i+1$-st row we see that the determinant we are looking for is equal to the determinant of

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 \\ 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ .. & .. & .. & \dots & .. & .. & .. \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

By swapping rows the last determinant can be brought into the lower diagonal form:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ .. & .. & .. & \dots & .. & .. & .. \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 \\ 1 & 2 & 3 & \dots & n-2 & n-1 & n \end{pmatrix}.$$

The determinant of this matrix is $n$. Note that we need $(n-1)+(n-2)+...+2+1=\frac{n(n-1)}{2}$ swaps to achieve this lower diagonal form. Thus, the value of the original determinant is

$$(-1)^{n(n-1)/2}\cdot n.$$