LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (2/16/2017 - 2/22/2017)

Problem: Let A be a matrix with $A^3 = 0$. Does there exist a matrix B such that $e^B = I + A$? Justify your claim. Here I denotes the identity matrix, and e^B stands for

$$e^B = I + B + \tfrac{1}{2}B^2 + \tfrac{1}{6}B^3 + ... + \tfrac{1}{n!}B^n + ...$$

Solution: Yes, the matrix B is given by $A - \frac{1}{2}A^2$; the formula for B is inspired by the expression $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - .....$

In the case of $A = 0$ we immediately see that $B = 0$ works; in the case when $A^2 = 0$ we see that $B = A$ works. We first investigate the situation when the Jordan normal form of A is

$$A' = P^{-1}AP = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix $B = A - \frac{1}{2}A^2$ is then similar (with the same transition matrix!) to

$$B' = P^{-1}BP = \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Observe that

$$(B')^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (B')^3 = 0.$$

Thus,

$$e^{B'} = I + B' + \tfrac{1}{2}(B')^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = I + A'$$

and

$$e^B = Pe^{B'}P^{-1} = P(I + A')P^{-1} = I + A.$$

The situation when A breaks into several nilpotent Jordan normal blocks is a direct consequence of the three cases addressed above.