LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (2/9/2017 - 2/15/2017)

Problem: You have 15 cookie jars containing 1, 2, 3, ..., 15 cookies respectively. You give Paul the permission to take any subset of the jars, and remove the exact same number of cookies from each of the jars he selected. What is the minimum number of moves in which Paul can empty out all the cookie jars? Justify your claim.

Solution: The smallest number of moves is 4.

Here is a way for Paul to clean out all the jars in 4 moves:

- He first removes 8 cookies from each of the jars 8 – 15;
- He then takes 4 cookies from jars 4 – 7 and 12 – 15;
- After that he takes 2 cookies from jars 2, 3, 6, 7, 10, 11, 14, 15;
- And in the last move he cleans up all the remaining cookies.

$$\begin{aligned}(1,2,...,15) &\longrightarrow (1,2,....7,0,1,2,...,7) \longrightarrow (1,2,3,0,1,2,3,0,...,3)\\ &\longrightarrow (1,0,1,...,0,1) \longrightarrow (0,...,0).\end{aligned}$$

To see that this is the most optimal strategy let $\vec{a_i} = (a_{1,i}, a_{2,i}, ..., a_{15,i})$ denote the numbers of cookies left in jars 1 – 15 and let C_i denote the number of distinct entries in $\vec{a}_i$ after the ith move. Note that $C_i \leq 2C_{i+1}$ because entries of $\vec{a}_i$ accounted for in C_i but not in C_{i+1} are obtained by bumping up a subset of those accounted for in C_{i+1} by a fixed amount. It thus follows that

$$C_0 \leq 2^n C_n.$$

At the termination of the process we have $C_n = 1$ and so the number n of steps needed for the process to terminate must satisfy

$$15 \leq 2^n.$$

It follows that $n \geq 4$.