LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (2/2/2017 - 2/8/2017)

Problem: Suppose a_1, a_2, ..., a_{2017} and b_1, b_2, ..., b_{2017} are two permutations of the set of numbers 1, 2, 3,..., 2017. Find, with proof, the minimum value of

$$a_1b_1 + a_2b_2 + ... + a_{2017}b_{2017}.$$

Solution: The minimum value is

$$1 \cdot 2017 + 2 \cdot 2016 + 3 \cdot 2015 + ... + 2016 \cdot 2 + 2017 \cdot 1 = 1\,369\,657\,969.$$

Without loss of generality we may assume that (in that particular order) we have

$$(a_1, a_2, ..., a_{2017}) = (1, 2, ..., 2017).$$

The claim is that the minimum value of the expression $a_1b_1 + ... + a_{2017}b_{2017}$ is then obtained for

$$(b_1, b_2, ..., b_{2017}) = (2017, 2016, ..., 1).$$

Suppose the opposite. Then there exists a pair on integers $1 \leq i < j \leq 2017$ such $b_i < b_j$. Let b'_1, b'_2, ..., b'_{2017} be the permutation obtained from b_1, b_2, ..., b_{2017} by swapping b_i and b_j. The difference between $a_1b_1 + ... + a_{2017}b_{2017}$ and $a_1b'_1 + ... + a_{2017}b'_{2017}$ is then equal to

$$a_ib_i + a_jb_j - a_ib'_i - a_jb'_j = ib_i + jb_j - ib_j - jb_i = (i - j)(b_i - b_j) > 0.$$

In other words, the value of $a_1b'_1 + ... + a_{2017}b'_{2017}$ would be smaller than $a_1b_1 + ... + a_{2017}b_{2017}$. Contradiction.