LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (1/19/2017 - 1/25/2017)

Problem: What is the maximum number of points on a circle of radius 1 such that the distance between any two of the points is strictly greater than $\sqrt{2}$? Justify your claim.

Solution: The maximum number is 3.

The existence of a triple of points with said property is clear from the example of $(1,0)$, $(-1/2, \sqrt{3}/2)$, $(-1/2, -\sqrt{3}/2)$ in the coordinate plane. Now suppose there existed four distinct points A_1, A_2, A_3 and A_4 on the unit circle with $d(A_i, A_j) > \sqrt{2}$ for all $1 \leq i \neq j \leq 4$. Without loss of generality we may assume that $A_1 = (1,0)$. Let us denote

$$A_2 = (a_{21}, a_{22}), \quad A_3 = (a_{31}, a_{32}), \quad A_4 = (a_{41}, a_{42}).$$

In algebraic terms the conditions that

$$d(A_1, A_2)^2 > 2, \quad d(A_1, A_3)^2 > 2, \quad d(A_2, A_3)^2 > 2$$

are equivalent to

$$a_{21} < 0, \quad a_{31} < 0, \quad a_{21}a_{31} + a_{22}a_{32} < 0,$$

respectively. Since $a_{21}a_{31} > 0$, we see that a_{22} and a_{32} must carry opposite $\pm$ signs. Using the point A_4 in place of the point A_3 allows us to conclude that $a_{41} < 0$ and that a_{22} and a_{42} must carry opposite $\pm$ signs. Consequently, a_{32} and a_{42} must carry the same $\pm$ sign and we have

$$d(A_3, A_4)^2 = (a_{31}^2 + a_{32}^2) + (a_{41}^2 + a_{42}^2) - 2(a_{31}a_{41} + a_{32}a_{42}) < 2.$$

Contradiction.