LEWIS AND CLARK COLLEGE
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SOLUTION OF THE PUZZLE OF THE WEEK

(11/16/2016 - 11/22/2016)

Problem: Suppose $X : \mathbb{N} \to \mathbb{R}$ is a random variable with $E[X^2] = 1$ and $E[X^4] = 2$. (Here $E[Y]$ denotes the expectation of the random variable Y.) Determine the largest possible value of $E[X^3]$, and justify your claim.

Solution: The maximum value is $\sqrt{2}$.

Let p denote the probability density function on $\mathbb{N}$. For simplicity let us use $x_n = X(n)$ and $p_n = p(n)$. We then have

$$E[X^2] = \sum_{n=1}^{\infty} x_n^2\, p_n = 1 \quad E[X^4] = \sum_{n=1}^{\infty} x_n^4\, p_n = 2. \tag{1}$$

By using the Cauchy-Schwarz Inequality for the weighted dot-product $(\vec{u}, \vec{v}) \mapsto \sum u_n v_n p_n$ we get

$$E[X^3] = \sum_{n=1}^{\infty} x_n^3\, p_n = \sum_{n=1}^{\infty} (x_n \cdot x_n^2)\, p_n \le \left(\sum_{n=1}^{\infty} x_n^2\, p_n \right)^{1/2} \cdot \left(\sum_{n=1}^{\infty} x_n^4\, p_n \right)^{1/2} = \sqrt{2}.$$

Furthermore, the equality is achieved when X^2 and X^4 are proportional:

$$x_n^2 = \lambda x_n^4 \ \text{ for all } \ n \in \mathbb{N}.$$

It follows from (1) that $1 = 2\lambda$, i.e $\lambda = \frac{1}{2}$. From here we see that non-zero values x_n of the most optimal X are all equal to $\sqrt{2}$ and that $E[X^3] = \sqrt{2} \cdot E[X^2] = \sqrt{2}$.