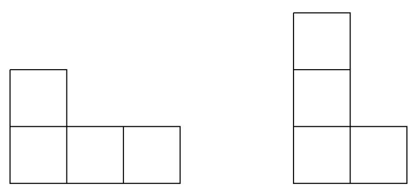
LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

SOLUTION OF THE PUZZLE OF THE WEEK

(10/5/2016 - 10/11/2016)

Problem: Consider the two kinds of L-shaped figures made out of 4 unit squares; see the picture below.

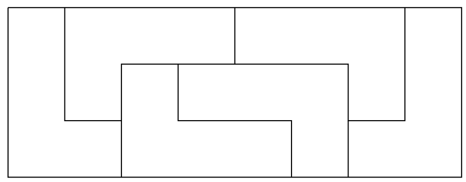


Let S be the set of all pairs (m, n) of integers $m, n > 1$ for which $m \times n$ rectangular grids can be tiled with such L-shaped pieces. Find, with proof, the set S.

Solution: The set S consists of pairs (m, n) for which mn is divisible by 8.

We first show that if the tiling exists then mn is divisible by 8. To see this, first observe that each L-piece has area of 4, implying that the area mn of the whole rectangle must be divisible by 4. Thus at least one of m and n is even. Suppose that the number of rows, m, is even. Color the rows of the rectangle alternately into black and white. The whole rectangular grid then has $\frac{1}{2}mn$ black pieces. On the other hand, each L-shape in the tiling covers an odd number of black pieces. Since $\frac{1}{2}mn$ is even, there must be an even number of L-shaped tiles. An even amount of L-shaped tiles covers an even multiple of 4, area-wise. So, mn must be divisible by 8.

Conversely, suppose that mn is divisible by 8. If both m and n are even then at least one of them is divisible by 4 making the whole table breakable into 4×2 blocks. In this case our claim follows from the fact that each 4×2 block can be tiled by 2 of the L-shapes. It remains to investigate the case when one m and n (without loss of generality, m) is odd while the other number (without loss of generality, n) is divisible by 8. In that case we cover the top $3 \times n$ strip by $n/8$ configurations which looks like the one below and



the remaining $(m-3) \times n$ grid as discussed in the case of even m and n.