LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (4/20/2016 - 4/26/2016)

Problem: Find all rational numbers r for which $\log_2 r$ is also rational. Justify your claim.

Solution: r has to be an integer power of 2.

Certainly, $r = 1$ is one such number. To find all other examples it suffices to find rational numbers $r > 1$ for which $\log_2 r$ is rational. Indeed, if $r > 1$ is one such number so is $1/r$ and vice versa. Let $r = p/q > 1$ with $\mathrm{GCD}(p, q) = 1$ be such that $\log_2 r = m/n$ is rational; we may assume $m, n > 0$. It follows that

$$2^{m/n} = p/q \quad \text{and} \quad q^n 2^m = p^n.$$

Since $m, n > 0$ we see that $2 \big| p$. As $\mathrm{GCD}(p, q) = 1$ we know that q is odd. Let us decompose: $p = 2^k l$ where $k > 0$ is an integer and where l is odd. Inserting into the above produces

$$q^n 2^{m-kn} = l^n.$$

Since both q and l are odd we must have $m - kn = 0$. This means that m/n is an integer and that (therefore) r is the power of 2.