LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (4/13/2016 - 4/19/2016)

Problem: The series

$$\sum_{n=0}^{\infty} \frac{2^n}{1+x^{2^n}} = \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \frac{8}{1+x^8} + \dots$$

converges for $|x| > 1$. Find, with proof, the expression for the sum of the series.

Solution: The series sums to $\frac{1}{x-1}$.

Observe that for $|x| > 1$ we have $|x^{-2^n}| < 1$; hence the geometric series expansion produces:

$$\begin{aligned}\frac{2^n}{1+x^{2^n}} =& \frac{2^n}{x^{2^n}} \cdot \frac{1}{1+x^{-2^n}} = \frac{2^n}{x^{2^n}} \left(1 - x^{-2^n} + (x^{-2^n})^2 - (x^{-2^n})^3 + \dots\right) \\ =& 2^n(x^{-2^n}) - 2^n(x^{-2\cdot 2^n}) + 2^n(x^{-3\cdot 2^n}) - \dots. = \sum_{k=1}^{\infty} (-1)^{k-1} 2^n x^{-k\cdot 2^n}.\end{aligned}$$

Note that these expansions are in fact absolutely convergent, so the value of the sum

$$\sum_{n=0}^{\infty} \frac{2^n}{1+x^{2^n}} = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} (-1)^{k-1} 2^n x^{-k\cdot 2^n}$$

is independent of the order of summation. We perform the summation by gathering terms with the same exponent $-N = -k \cdot 2^n$ on the x variable. Let m be the highest non-negative integer with $2^m \,\big|\, N$. Then the term x^{-N} appears $1+m$ times in the summation, and the corresponding coefficient is:

$$-1 - 2 - \dots - 2^{m-1} + 2^m = 1.$$

It follows that

$$\sum_{n=0}^{\infty} \frac{2^n}{1+x^{2^n}} = \sum_{N=1}^{\infty} x^{-N} = \frac{x^{-1}}{1-x^{-1}} = \frac{1}{x-1}.$$