LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

# PUZZLE OF THE WEEK (4/6/2016 - 4/12/2016)

**Problem:** Let $a_1, a_2, ..., a_n$ be positive numbers with $a_1 + a_2 + ... + a_n = 1$. Find, with proof, the minimum value of

$$\sum \frac{1}{a_{i_1} + a_{i_2} + ... + a_{i_k}}$$

where the summation goes over all choices of $i_1, i_2, ..., i_k \in \{1, 2, ..., n\}$ with $i_1 < i_2 < ... < i_k$.

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**Solution:** The minimum is $\frac{n}{k}\binom{n}{k}$.

To see this, one just needs to apply the inequality between the harmonic and the arithmetic means. There are $\binom{n}{k}$ terms in our summation and so

$$\frac{\binom{n}{k}}{\sum \frac{1}{a_{i_1}+a_{i_2}+...+a_{i_k}}} \leq \frac{\sum (a_{i_1} + a_{i_2} + ... + a_{i_k})}{\binom{n}{k}}.$$

In the summation on the right each $a_i$ appears $\binom{n-1}{k-1}$ times and thus the value on the right hand side of the inequality is

$$\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}.$$

The fact that

$$\frac{n}{k}\binom{n}{k} \leq \sum \frac{1}{a_{i_1} + a_{i_2} + ... + a_{i_k}}$$

is now immediate. To see that this minimum value can actually be reached we use the choice of

$$a_1 = a_2 = ... = a_n = \frac{1}{n}.$$