LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (3/9/2016 - 3/15/2016)

Problem: Find, with proof, all integer solutions (x, y, u, v) of the equation

$$x^2 + y^2 = 3(u^2 + v^2).$$

Solution: The only solution is $x = y = u = v = 0$.

To see this suppose the opposite: that there is a non-trivial solution. Then there are solutions with $x, y, u, v \geq 0$ and $x + y + u + v > 0$. Amongst those solutions find one for which the total sum $x + y + u + v$ is minimum possible. (Such a solution exists because subsets of natural numbers always reach their minimum.) From now on we refer to this "minimal" solution as (x_*, y_*, u_*, v_*).

We first observe that $(x_*, y_*) \neq (0, 0)$; otherwise we would have $0 = 3(u_*^2 + v_*^2)$ and consequently $(x_*, y_*, u_*, v_*) = (0, 0, 0, 0)$. In particular, we now know that

$$x_* + y_* > 0.$$

Also note that $x_*^2 + y_*^2$ is divisible by 3. Since $(3k \pm 1)^2 = 3(3k^2 \pm 2k) + 1$ we see that perfect squares can only leave remainders of 0 and 1 after division by 3. So, the only way for $x_*^2 + y_*^2$ to be divisible by 3 is if both x_*^2 and y_*^2 are divisible by 3. In other words, we must have

$$x_* = 3x_{**} \quad \text{and} \quad y_* = 3y_{**}$$

for some non-negative integers x_{**} and y_{**}. Inserting this information into $x_*^2 + y_*^2 = 3(u_*^2 + v_*^2)$ produces

$$9(x_{**}^2 + y_{**}^2) = 3(u_*^2 + v_*^2) \quad \text{i.e} \quad u_*^2 + v_*^2 = 3(x_{**}^2 + y_{**}^2).$$

In particular, $(u_*, v_*, x_{**}, y_{**})$ is another non-negative integer solution of our equation. By the minimality of (x_*, y_*, u_*, v_*) we have

$$u_* + v_* + x_{**} + y_{**} \geq x_* + y_* + u_* + v_* \quad \text{i.e} \quad \frac{x_*}{3} + \frac{y_*}{3} \geq x_* + y_*.$$

The latter is impossible since $x_* + y_* > 0$. This completes our proof that $(0, 0, 0, 0)$ is the only integer solution of our equation.