LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (3/2/2016 - 3/8/2016)

Problem: Suppose that OA, OB, OC and OD are four non-coplanar rays in (three-dimensional) space such that the six angles formed by the rays are mutually congruent. What are the measures of these angles? Justify your claim.

Solution: The angles are arccos(1/3) each.

Without loss of generality we may assume that $|OA| = |OB| = |OC| = |OD| = 1$. By Side-Angle-Side we conclude that the six triangles ΔAOB, ΔAOC, etc are all congruent. In particular, it follows that

$$|AB| = |AC| = |AD| = |BC| = |BD| = |CD|.$$

Consequently, $ABCD$ is a regular tetrahedron and O is its center. Since O is the center of the tetrahedron we have

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = \vec{0}.$$

Applying the dot product to both sides of this identity produces

$$\overrightarrow{OA} \cdot \overrightarrow{OA} + \overrightarrow{OB} \cdot \overrightarrow{OA} + \overrightarrow{OC} \cdot \overrightarrow{OA} + \overrightarrow{OD} \cdot \overrightarrow{OA} = 0 \text{ i.e. } 1 + 3\cos(x) = 0,$$

where x denotes the unknown angle. Solving for x proves our claim.