LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (2/17/2016 - 2/23/2016)

Problem: One vertex of a rectangle is found at a point A. Two other vertices, not belonging to the same side, lie on two given lines which form a right angle at a point O. Where is the fourth vertex of the rectangle located? Justify your claim. (You may assume that A is not on the two given lines, and that these lines and the point A lie in the same plane.)

Solution: The fourth vertex of the rectangle can be located anywhere along the line through O which is orthogonal to OA.

Let $ABCD$ be a rectangle fitting the descriptions of the problem, with vertices B and C on the two given orthogonal lines. Since angles $\angle BAC$, $\angle BDC$ and $\angle BOC$ are all right, we know that A, B, C, D and O all lie on the circle with diameter BC. On the other hand, the midpoint of BC is the midpoint of AD and thus the angle $\angle AOD$ is also right. (It suffices to study the case when $O \neq D$.) This proves that D has to lie on the line through O which is orthogonal to OA.

Conversely, let D be any point on the line through O which is orthogonal to OA. Consider the circle $\mathcal{C}$ with diameter AD, and its other two intersections points with the given lines: B and C. (If one of the lines is tangential to $\mathcal{C}$ then take $B = O$ or $C = O$.) Note that the angle $\angle BOC$ is right; consequently, the circle $\mathcal{C}$ is centered at the midpoint of BC. (The latter remains to hold even if $B = O$ or $C = O$.) It now follows that $\angle BAC$ and $\angle BDC$ are both right, and that $ABCD$ is a rectangle.