LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (2/10/2016 - 2/16/2016)

Problem: A square is intersected by n lines, no three of which go through the same point. It is known that each of these lines divides the square into two trapezoids whose areas are in a $2 \div 3$ ratio. What is the largest value of n? Justify your claim.

Solution: The maximum value of n is 8.

Let the the points P, Q, R and S be the midpoints of the sides AB, BC, CD and DA of the square $ABCD$. Let K and L be the points on the line segment PR such that

$$PK \div KR = 2 \div 3, \quad \text{and} \quad PL \div LR = 3 \div 2.$$

Likewise, let M and N be the points on the line segment QS such that

$$QM \div MS = 2 \div 3, \quad \text{and} \quad QN \div NS = 3 \div 2.$$

Now let ℓ be any line which intersects the square $ABCD$ into a $2 \div 3$ area ratio. First assume that ℓ intersects AB, QS and CD. Let X be the intersection of ℓ and QS. Since the area of the trapezoid with base AD and the area of the trapezoid with base BC are in a $2 \div 3$ or $3 \div 2$ ratio, it follows that

$$SX \div XQ = 2 \div 3 \ \text{ or } \ SX \div XQ = 3 \div 2;$$

hence $X = N$ or $X = M$. In other words, ℓ must pass through either N or M. If on the other hand, the line ℓ intersects BC, PR and AD the same argument can be employed to conclude that ℓ passes through K or L.

We now have n lines such that each line passes through at least one of K, L, M and N and such that no three lines pass through the same point. By the Pigeonhole Principle we must have $n \leq 8$.

On the other hand, note that each of the 8 lines

$$PN,\ PM,\ RN,\ RM,\ QL,\ QK,\ SL,\ SK$$

divide the square $ABCD$ in a $2 \div 3$ ratio, area-wise. Furthermore, no three of these lines pass through the same point, as evidenced by the fact that the parallelograms $PMRN$ and $QLSK$ have no vertices in common. Thus, $n = 8$.