LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (1/27/2016 - 2/2/2016)

Problem: Two doctors, Alice and Bob, are asked to examine the same group of n patients. Each appointment would be 15 minutes long. The doctors would start at the same time and take breaks at the same time. In how many ways can the patients (and their appointments) be arranged so that all $2n$ appointments are completed in $n/4$ examination hours?

Solution: There are $(n!)^2\left(1-\frac{1}{1}+\frac{1}{2!}-...+(-1)^j\frac{1}{j!}+...+(-1)^n\frac{1}{n!}\right)$ desirable arrangements.

Let $a_1, a_2, ..., a_n$ be Alice's appointments and $b_1, b_2, ..., b_n$ be Bob's appointments. Note that $b_1, b_2, ..., b_n$ is simply a permutation of $a_1, a_2, ..., a_n$. For each possible arrangement $a_1, a_2, ..., a_n$ (and there are $n!$ such arrangements) we need to determine the number of permutations $b_1, b_2, ..., b_n$ of $a_1, a_2, ..., a_n$ in which $b_j \neq a_j$ for all $1 \leq j \leq n$.

Let X_j (for a fixed $1 \leq j \leq n$) denote the set of permutations where $b_j = a_j$. The idea here is to use the Inclusion-Exclusion Principle to determine

$$n! - |X_1 \cup X_2 \cup ... \cup X_n|.$$

Recall that the Inclusion-Exclusion Principle states

$$|X_1 \cup X_2 \cup ... \cup X_n| = \sum (-1)^{l-1}|X_{i_1} \cap X_{i_2} \cap ... \cap X_{i_l}|,$$

where the summation goes over all $l \geq 1$, and over all possible subsets $\{i_1, i_2, ..., i_l\} \subseteq \{1, 2, ..., n\}$.

To apply this principle note the following cardinalities (the number of elements):

$$|X_j| = (n-1)!, \quad |X_j \cap X_k| = (n-2)!, \quad |X_j \cap X_k \cap X_l| = (n-3)!, ...$$

Since there are $\binom{n}{l}$ possibilities for $|X_{i_1} \cap X_{i_2} \cap ... \cap X_{i_l}|$ we see that $|X_1 \cup X_2 \cup ... \cup X_n|$ is

$$\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + ... + (-1)^{j-1}\binom{n}{i}(n-i)! + ... + (-1)^{n-1}\binom{n}{n}0!$$

To summarize: it follows from the Inclusion-Exclusion Principle that the number of permutations $b_1, b_2, ..., b_n$ of $a_1, a_2, ..., a_n$ in which $b_j \neq a_j$ for all $1 \leq j \leq n$ is

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - .... + (-1)^{j}\binom{n}{i}(n-i)! + ... + (-1)^{n}\binom{n}{n}0!$$

The above expression can be simplified into

$$n!\left(1 - \tfrac{1}{1} + \tfrac{1}{2!} - ... + (-1)^j \tfrac{1}{j!} + ... + (-1)^n \tfrac{1}{n!}\right),$$

so that the final answer is

$$(n!)^2\left(1 - \tfrac{1}{1} + \tfrac{1}{2!} - ... + (-1)^j \tfrac{1}{j!} + ... + (-1)^n \tfrac{1}{n!}\right).$$