LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

PUZZLE OF THE WEEK (1/20/2016 - 1/27/2016)

Problem: Edges of a regular hexagon are to be colored in blue or red. We consider two colorings to be the same if there is a rotation, reflection or a composition of rotations and reflections which converts one coloring to the other. Determine the number of distinct colorings of the hexagon. Justify your answer.

Solution: The answer is 13.

There are direct ways of arriving at this answer, but there is also a bigger combinatorics lesson hiding here: a lesson on Pólya-Burnside Theorem.

Theorem 1 *Let S be a finite collection of objects and let $G = \{g_1, g_2, ..., g_n\}$ be a group of symmetries of S (identity symmetry included). For a symmetry g let S^g denote the set of objects in S which are unchanged by g. The number of objects in S which are distinguishable relative to the group of symmetries G is given by*

$$\frac{1}{|G|}\left(|S^{g_1}| + |S^{g_2}| + ... + |S^{g_n}|\right),$$

where $|\cdot|$ denotes the number of elements of a set.

Here is how we can apply the theorem to our situation. Observe that the total set of colorings S has $2^6 = 64$ elements, while the group of symmetries G has 12 elements: 6 rotations and 6 reflections. (Note that 0^o rotation i.e identity is included here.) There are only 2 colorings which are left unchanged by $\pm 60^o$-degree rotations, 4 that are left unchanged by $\pm 120^o$-degree rotations, and 8 that are left unchanged by the 180^o-degree rotation. For each reflection there are 4 colorings which are left unchanged by it. So the Pólya-Burnside Theorem gives us

$$\frac{1}{12}\left(64 + 2 \cdot 2 + 2 \cdot 4 + 8 + 12 \cdot 4\right) = \frac{1}{12} \cdot 132 = 13$$

distinguishable colorings.