Distance comparison and the Dirichlet problem for curve shortening flow in convex domains

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Summary

The curve shortening flow (CSF), which evolves a curve in the normal direction with velocity proportional to its curvature, has been explored extensively for curves in the Euclidean plane. It is known that embedded, closed curves shrink to round points in finite time. The case of curves with fixed endpoints remains largely unexamined. We prove results for the evolution of a curve under curve shortening flow on the plane and the sphere.

Curve Shortening Flow

Let $F : [a, b] \to \Sigma$ parameterize a smooth curve. The first variation of L, the length of the curve, defines the curvature vector K. We consider the Curve Shortening Flow evolution equation

$$\partial_t F = K,$$

subject to a Dirichlet boundary condition

$$F(\cdot, t)\big|_{\partial I} = F(\cdot, 0)\big|_{\partial I}.$$

Let T be the unit tangent vector; the CSF corresponds to

$$(\partial_t F)^i = (\nabla_T \nabla_T F)^i = \partial^2_{ss} F^i + \Gamma^i_{kl} T^k T^l$$

where s is the arc-length parameter, and Γ is the Christoffel symbol determined by the metric on Σ.

On $\mathbb{R}^2$, in Cartesian coordinates the CSF is given by

$$\partial_t \begin{pmatrix} F^x \\ F^y \end{pmatrix} = \partial^2_{ss} \begin{pmatrix} F^x \\ F^y \end{pmatrix}.$$

On $\mathbb{S}^2$, the CSF is given in spherical coordinates (θ, ϕ) by

$$\partial_t \begin{pmatrix} F^\theta \\ F^\phi \end{pmatrix} = \partial^2_{ss} \begin{pmatrix} F^\theta \\ F^\phi \end{pmatrix} + \begin{pmatrix} -2\tan\phi\, \partial_s F^\theta \partial_s F^\phi \\ \cos\phi \sin\phi \left(\partial_s F^\theta\right)^2 \end{pmatrix},$$

where $(\theta, 0)$ is a point on the equator.

Geometry of the Curve Shortening Flow

Let N be a unit normal vector, and let κ be the scalar curvature with respect to N: $\kappa = \langle K, N \rangle$. The Frenet equations take the following form:

$$\nabla_T T = \kappa N \quad \text{and} \quad \nabla_T N = -\kappa T.$$

Many results for curve shortening flow in the plane from Gage-Hamilton [2] can be generalized to arbitrary 2-manifolds.

$$\frac{dL}{dt} = -\int \kappa^2 ds.$$

$$\nabla_{\frac{d}{dt}} T = T(\kappa) N \quad \text{and} \quad \nabla_{\frac{d}{dt}} N = -T(\kappa) T.$$

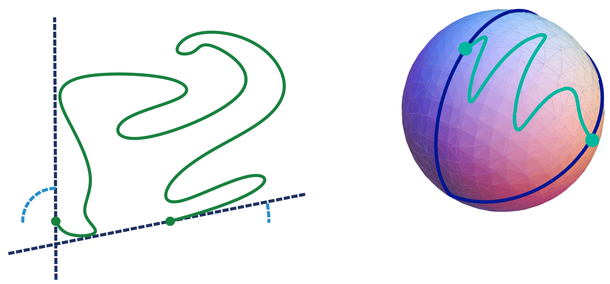
Most importantly, we have the following evolution equation for curvature:

$$\frac{d}{dt}\kappa = \nabla_T \nabla_T \kappa + \kappa^3 + \kappa R(N, T, T, N);$$

$R(N, T, T, N)$ is called the "sectional curvature" of Σ.

Main Results

- We can flow a curve F so long as the curvature is defined.
- We use a distance comparison, together with control near the endpoints, to show that curves in convex regions of the plane and sphere do not form singularities.
- Therefore the flow exists for all time and the curves approach the geodesic connecting the (fixed) endpoints.



Examples of curves in convex regions of the plane and sphere.

Distance Comparisons

Let $D(p, q)$ be the geodesic distance between two points p, q on the curve, and $L(p, q)$ be the distance along the curve. Following [3], we look at the ratio D/L as a measure of the straightness of the curve. Note that $D/L \equiv 1$ when F is a geodesic. We prove that, in $\mathbb{R}^2$ or $\mathbb{S}^2$, if D/L attains a local interior minimum at (p, q) at time T, then

$$\frac{d}{dt}\frac{D}{L}(p, q; T) \geq 0$$

with equality if and only if F is a geodesic. Our proof when $\Sigma = \mathbb{R}^2$ is a modification of that in [3].

Control Near the Endpoints

In a neighborhood of each endpoint it is possible to choose coordinates so that the curve is given by a graph. Applying work of Angenent [1] we can show that the curve remains an embedded graph and does not form singularities in the chosen neighborhood.

Thanks

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[1] Sigurd Angenent. Nodal properties of solutions of parabolic equations. *Rocky Mountain J. Math.*, 21(2):585–592, 1991. Current directions in nonlinear partial differential equations (Provo, UT, 1987).

[2] M. Gage and R. S. Hamilton. The heat equation shrinking convex plane curves. *J. Differential Geom.*, 23(1):69–96, 1986.

[3] Gerhard Huisken. A distance comparison principle for evolving curves. *Asian J. Math.*, 2(1):127–133, 1998.