LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

SOLUTION OF THE PUZZLE OF THE WEEK

(11/2/2016 - 11/8/2016)

Problem: Suppose that for some continuously differentiable, decreasing function $f(x)$ defined on the real number line the series $\sum_{n=1}^{\infty} f(n^2)$ converges. Is it necessarily the case that the series $\sum_{n=1}^{\infty} \frac{f(n)}{\sqrt{n}}$ converges? Justify your claim.

Solution: Yes, under the given circumstances the series $\sum_{n=1}^{\infty} \frac{f(n)}{\sqrt{n}}$ must be convergent.

Since the series $\sum f(n^2)$ converges we have $f(n^2) \to 0$ as $n \to \infty$. As $f(x)$ is decreasing we see that $f(x)$ is positive and satisfies $\lim_{x\to\infty} f(x) = 0$. It now follows from the Integral Test that

$$\int_1^{\infty} f(x^2)\, dx < \infty. \tag{1}$$

Also note that the Integral Test applies to series $\sum \frac{f(x)}{\sqrt{x}}$; indeed, the function $\frac{f(x)}{\sqrt{x}}$ is positive, decreasing:

$$\frac{d}{dx}\left(\frac{f(x)}{\sqrt{x}}\right) = \frac{\frac{df}{dx}}{\sqrt{x}} - \frac{f(x)}{2\sqrt{x^3}} < 0$$

and converges to zero as $x \to \infty$. Thus, to establish the convergence / divergence of the series $\sum \frac{f(x)}{\sqrt{x}}$ it suffices to study the improper integral $\int_1^{\infty} \frac{f(x)}{\sqrt{x}}\, dx$. The latter converges because of (1) and the change of variables $x = \sqrt{u}$:

$$\int_1^{\infty} \frac{f(u)}{\sqrt{u}}\, du = 2\int_1^{\infty} f(x^2)\, dx < \infty.$$