LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

SOLUTION OF THE PUZZLE OF THE WEEK
(10/26/2016 - 11/1/2016)

Problem: Find all polynomials $P(x)$ with real coefficients for which

$$P(3-2x) = 4P(x)$$

for all real x.

Solution: The only such polynomials take the form of

$$P(x) = C(x-1)^2, \quad C \text{ a real number.}$$

Suppose r was a root of $P(x)$; then so would be $3-2r$. If $r \neq 1$ then the sequence

$$r_0 = r, \quad r_1 = 3-2r = -2(r-1)+1,$$
$$r_2 = 3-2r_1 = 4(r-1)+1, \quad r_3 = 3-2r_2 = -8(r-1)+1, \ldots$$

i.e. the sequence $r_n = (-2)^n(r-1)+1$, has infinitely many distinct terms all of which serve as roots of P. Since polynomials can only have finitely many roots, it follows that $r=1$ is the only possible root of $P(x)$. In other words, $P(x)$ must take the form of

$$P(x) = C(x-1)^n$$

for some real number C and some non-negative integer n. In that case we have

$$P(3-2x) = 2^n C(x-1)^n = 2^n P(x),$$

and we see that $n=2$. Overall, we conclude that $P(x) = C(x-1)^2$.