LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

SOLUTION OF THE PUZZLE OF THE WEEK

(10/19/2016 - 10/25/2016)

Problem: Let n be a positive integer. Find, with proof, the value of the sum

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots.$$

Solution: The value is $2^{n/2}\cos(n\pi/4)$.

To see this observe that, by the Binomial Theorem, the quantity in question is the real part of the expression $(1+i)^n$:

$$(1+i)^n = \left(1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots.\right) + i\left(\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \ldots.\right).$$

On the other hand, this power is most efficiently computed using the Euler Formula:

$$(1+i)^n = (\sqrt{2}e^{i\pi/4})^n = 2^{n/2}\left(\cos(n\pi/4) + i\sin(n\pi/4)\right).$$

Thus, we have

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots. = 2^{n/2}\cos(n\pi/4)$$

as well as

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \ldots. = 2^{n/2}\sin(n\pi/4).$$