LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

SOLUTION OF THE PUZZLE OF THE WEEK

(10/12/2016 - 10/18/2016)

Problem: Let $A_1A_2...A_n$ be a regular polygon inscribed in a circle of radius 1. Find, with justification, the maximum value of

$$|PA_1| \cdot |PA_2| \cdot ... \cdot |PA_n|$$

as P ranges over the circumcircle. (Here $|PA_k|$ denotes the length of the line segment PA_k.)

Solution: The maximum value is 2.

Place the vertices of the polygon on the unit circle in the complex plane so that

$$A_k = \exp(2\pi i \tfrac{k}{n}), \quad 1 \le k \le n.$$

If the point P corresponds to the complex number z then

$$\begin{aligned} |PA_1| \cdot |PA_2| \cdot ... \cdot |PA_n| &= \prod_{k=1}^{n} \left|\left(z - \exp(2\pi i \tfrac{k}{n})\right)\right| \\ &= \left|\prod_{k=1}^{n} \left(z - \exp(2\pi i \tfrac{k}{n})\right)\right| = |z^n - 1|, \end{aligned}$$

by virtue of the fact that $\exp(2\pi i \frac{k}{n})$ represent roots of the polynomial $z^n - 1$. By the Triangle Inequality we have

$$|z^n - 1| \le |z|^n + 1,$$

with equality reached only when z^n is a negative real number. Thus $z^n = -1$ and the maximum value of the said product is 2. It is reached when z is n-th root of -1:

$$z = \exp(\pi i \tfrac{2k+1}{n}).$$