LEWIS AND CLARK COLLEGE
Department of Mathematical Sciences

SOLUTION OF THE PUZZLE OF THE WEEK

(9/28/2016 - 10/4/2016)

Problem: Is it true that every positive integer can be represented as

$$\pm 1^2 \pm 2^2 \pm \ldots \pm n^2$$

for some positive integer n and some choice of $\pm$ signs? Justify your claim.

Solution: Yes, every positive integer is expressible in such a form.
We prove this by strong induction. In the base cases we have:

$$1 = +1^2, \quad 2 = -1^2 - 2^2 - 3^2 + 4^2, \quad 3 = -1^2 + 2^2, \quad 4 = -1^2 - 2^2 + 3^2.$$

Suppose that for some $N \geq 4$ all the integers k with $1 \leq k \leq N$ can be represented as a sum $\sum_{i=1}^{n} \pm i^2$ (for some suitable n and a suitable choice of $\pm$ signs). It suffices to show that $N+1$ can also be represented in such a form. Consider the number $N-3$; clearly $1 \leq N-3 \leq N$ and the inductive hypothesis applies to $N-3$:

$$N - 3 = \sum_{i=1}^{n} \pm i^2.$$

We now observe that

$$4 = (n+1)^2 - (n+2)^2 - (n+3)^2 + (n+4)^2.$$

Adding to the latter to the expression for $N-3$ produces the desired decomposition of $(N-3)+4 = N+1$. This completes our inductive proof.